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1. Koralov and Sinai Ch. 1, Problem 1

   A man's birthday is on March 1st. His father's birthday is on March 2nd. One of his grandfather's birthday is on March 3rd. How would you estimate the number of such people in the USA?

   Solution. First, establish the number of men in the United States, say n. Define sets U, V, W as the following

$$\begin{aligned} U &= \{x \mid x \text{ is a man born on March 1st}\} \\ V &= \{x \mid x \text{ is a man whose father's birthday is March 2nd}\} \\ W &= \{x \mid x \text{ is a man who has at least one grandfather born on March 3rd}\}. \end{aligned}$$

   Next, we need to establish the probability of a man being in one of the above sets. For the most part, the probability of a man being in each set will be the size of the set divided by n.

   The probability of the occurrence presented in the problem is then

$$\mathrm{P}[x \in U \cap V \cap W] = \mathrm{P}[x \in U] \cdot \mathrm{P}[x \in V] \cdot \mathrm{P}[x \in W]$$

   since the events are independent. Now, the number m of such men in the United States who satisfy the conditions of the problem are

$$m = \mathrm{P}[x \in U \cap V \cap W] \cdot n.$$

   In simple terms, $\mathrm{P}[x \in U] = \mathrm{P}[x \in V] = \frac{1}{365}$ and $\mathrm{P}[x \in W] = \frac{2}{365}$. Assuming that the number of men in the USA is roughly half of the 300 million total,

$$\begin{aligned} N &= 300,000,000 \cdot \frac{2}{365^3} \\ &\approx 6 \text{ such men} \end{aligned}$$

2. Koralov and Sinai Ch. 1, Problem 2

   Suppose that n identical balls are distributed randomly among m boxes. Construct the corresponding space of elementary outcomes. Assuming that each ball is placed in a random box with equal probability, find the probability that the first box is empty.

   Solution. Define the space of elementary outcomes to be Ω,

$$\begin{aligned} \Omega = \Big\{ & n \text{ balls in 1 box} \binom{m}{1} \cdot 1^n \text{ ways}, n \text{ balls in 2 boxes} \binom{m}{2} \cdot (2^n - 1^n) \text{ ways}, \\ & \ldots \; n \text{ balls in } k \text{ boxes} \binom{m}{k} \cdot (k^n - \sum_{j=1}^{k-1} j^n) \text{ ways}, \\ & \ldots \; n \text{ balls in } m \text{ boxes} \binom{m}{m} \cdot (m^n - \sum_{j=1}^{m-1} j^n) \text{ ways} \Big\}. \end{aligned}$$

For brevity's sake, I hope the above construction will suffice. The size of the set Ω is m^n.

The probability that the first box remains empty can be quantified via a counting argument, i.e. where box 1 is excluded from the possible ways of filling the boxes. Assuming that each of the m^n possibilities occur with equal probability m^{-n}, the probability that any one box (and only one box) remains empty is then

$$p_0 = \binom{n}{0}\frac{(m-1)^{n-0}}{m^n}.$$

As a confirmation, I ran 5 monte carlo simulations of 10000 runs for the case when $n = 24$ and $m = 17$. Each of the 5 simulations had different initial seeds. The simulation average (mean over 5 trials) was ≈ 0.2312 while the analytic calculation is $p_0 \approx 0.2334$. □

3. Koralov and Sinai Ch. 1, Problem 3

   A box contains 90 good items and 10 defective items. Find the probability that a sample of 10 items has no defective items.

   Solution. The probability is established via the following counting argument,

$$\begin{aligned} \text{P[10 good samples picked from 100]} &= \left(\frac{90}{100}\right)\cdot\left(\frac{89}{99}\right)\cdot\dots\cdot\left(\frac{81}{91}\right) \\ &= \frac{(90!)^2}{100!80!} \\ &\approx 0.3305 \end{aligned}$$

4. Koralov and Sinai Ch. 1, Problem 4

   Let ξ be a random variable such that $\mathrm{E}|\xi|^m \le AC^m$ for some positive constants A and C, and all integers $m \ge 0$. Prove that $\mathrm{P}(|\xi| > C) = 0$.

   Proof. Let $W_{C,n} = \{\omega|\|\xi(\omega)| \ge C + \frac{1}{n}\} = \{\omega|\ |\xi(\omega)|^m \ge (C+\frac{1}{n})^m\}$. Then,

$$\begin{aligned} \mathrm{P}[|\xi| > C] &= \mathrm{P}\big[\xi \in \bigcup_{n=1}^{\infty} W_{C,n}\big] \\ &\le \sum_{n=1}^{\infty} \mathrm{P}\big[\xi \in W_{C,n}\big] \\ &\le \sum_{n=1}^{\infty} \frac{E|\xi|^m}{(C+\frac{1}{n})^m} \text{ by Chebyshev} \\ &\le \sum_{n=1}^{\infty} A\left(\frac{C}{C+\frac{1}{n}}\right)^m \\ &= 0 \end{aligned}$$

since the given inequality must hold for all m. Therefore,

$$\mathrm{P}[|\xi| > C] = 0$$

$\square$

5. Koralov and Sinai Ch. 1, Problem 5

   Suppose there are n letters addressed to n different people, and n envelopes with addresses. The letters are mixed and then randomly placed into the envelopes. Find the probability that at least one letter is in the correct envelope. Find the limit of this probability as $n \to \infty$.

   Solution. It seems best to tackle this problem with the Inclusion-Exclusion Principle. Consider the sets A_i, the sets of all possible outcomes where i letters are in the right place. Therefore, consider $W = \left|\sum_{m=1}^{n} A_m\right|$; that is, the number of configurations where at least one letter is in the right place. Now, by the inclusion-exclusion formula,

$$\begin{aligned} W &= \binom{n}{1}|A_1| - \binom{n}{2}|A_1 \cap A_2| + \binom{n}{3}|A_1 \cap A_2 \cap A_3| - \dots \\ &= \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}(n-j)! \\ &= \sum_{j=1}^{n}(-1)^{j-1}\frac{n!}{j!}. \end{aligned}$$

   Since the number of possible configurations of n letters in n envelopes is $n!$, the probability that at least one envelope is in the right place is

$$\begin{aligned} \mathrm{P}[\text{at least one envelope is in the right place}] &= \frac{W}{n!} \\ &= 1 - \sum_{j=0}^{n}\frac{(-1)^j}{j!}. \end{aligned}$$

   In the limit as $n \to \infty$, the probability takes the following nice form,

$$\begin{aligned} \mathrm{P}[\text{at least one envelope is in the right place}] &= 1 - \sum_{j=0}^{\infty}\frac{(-1)^j}{j!} \\ &= 1 - e^{-1} \\ &\approx 0.6321 \end{aligned}$$

6. Koralov and Sinai Ch. 1, Problem 7

   Find the mathematical expectation and the variance of a random variable with Poisson distribution with parameter λ.

Solution. The density function for the Poisson distribution is

$$p(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Therefore, the expected value of k is

$$\begin{aligned} \mathrm{E}[k] &= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda \end{aligned}$$

The variance is a bit trickier, but we can decompose the variance as the following:

$$\begin{aligned} \mathrm{Var}[k] &= \mathrm{E}[k^2] - \mathrm{E}[k]^2 \\ &= \mathrm{E}[k^2] - \mathrm{E}[k] + \mathrm{E}[k] - \mathrm{E}[k]^2 \\ &= \mathrm{E}[k(k-1)] + \mathrm{E}[k] - \mathrm{E}[k]^2 \end{aligned}$$

which is a nice trick allowing simple evaluation of the variance. The variance is

$$\begin{aligned} \mathrm{Var}[k] &= \mathrm{E}[k(k-1)] + \mathrm{E}[k] - \mathrm{E}[k]^2 \\ &= e^{-\lambda} \sum_{k=0}^{\infty} k(k-1)\frac{\lambda^k}{k!} + \lambda - \lambda^2 \\ &= e^{-\lambda}\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda - \lambda^2 \\ &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda \end{aligned}$$

7. Koralov and Sinai Ch. 1, Problem 8

   Draw the graph of the distribution function of random variable ξ taking values $x_1, \ldots, x_n$ with probabilities $\mathrm{p}_1, \ldots, \mathrm{p}_n$.

   Solution. The plot of the distribution function $F(x)$ is shown below. The key points to highlight are that the function is

   - $F(x_j)$ is monotonically increasing
   - $F(x_j) = \sum_{i<j} \mathrm{P}[x_i]$
   - $F(x_j) = 1$ when $j > n$

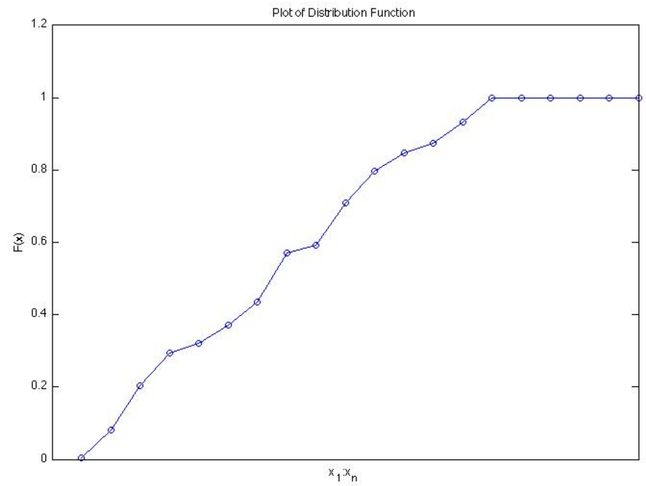

Figure 1: Plot of $F(x)$

8. Koralov and Sinai Ch. 1, Problem 9

   Prove that if F is the distribution function of the random variable ξ, then $\mathrm{P}(\xi = x) = F(x) - \lim_{\delta \downarrow 0} F(x - \delta)$.

   Proof. Assume WLOG that $\delta = \frac{1}{n}$. Define $C_x = \{\omega \in \Omega : \xi(\omega) = x\}$, $C_{x,n} = \big\{\omega \in \Omega : 0 \le x - \xi(\omega) \le \frac{1}{n}\big\}$. Clearly

$$\bigcap_{n=1}^{\infty} C_{x,n} = C_x.$$

   Then,

$$\begin{aligned} F(x) - F\big(x - \frac{1}{n}\big) &= \mathrm{P}[\xi \le x] - \mathrm{P}\big[\xi \le x - \frac{1}{n}\big] \\ &= \mathrm{P}\big[x - \frac{1}{n} < \xi \le x\big] \\ &= \mathrm{P}[C_{x,n}] \end{aligned}$$

   Now, take the limit as $n \to \infty$,

$$\begin{aligned} F(x) - \lim_{\delta \downarrow 0} F\big(x - \delta\big) = F(x) - \lim_{n \to \infty} F\big(x - \frac{1}{n}\big) &= \lim_{n \to \infty} \mathrm{P}[C_{x,n}] \\ &= \mathrm{P}\Big[\bigcap_{n=1}^{\infty} C_{x,n}\Big] \\ &= \mathrm{P}[C_x] \\ &= \mathrm{P}[\xi = x] \end{aligned}$$

$\square$

9. Koralov and Sinai Ch. 1, Problem 10

   A random variable ξ has density p. Find the density of $\eta = a\xi + b$ for $a, b \in \mathbb{R}$, $a \neq 0$.

   Solution. The action of $\eta = a\xi + b$ translates each interval where $P[\xi] > 0$ over by b and scales by a. Therefore, the density of η can be thought of as a translation and scaling of the argument to p that "moves" the interval where η is defined onto the interval where ξ is defined. Additionally, there needs to be a rescaling by $1/|a|$. The density of η is then, assuming ξ is defined on (m, n) and 0 elsewhere,

$$q(\eta) = \begin{cases} \frac{1}{|a|} p\left(\frac{\eta - b}{a}\right) & \eta \in (am + b, an + b) \\ 0 & \text{else} \end{cases}$$

10. Koralov and Sinai Ch. 1, Problem 11

   A random variable ξ has uniform distribution on $[0, 2\pi]$. Find the density of the distribution $\eta = \sin \xi$.

   Solution. The sin action takes the uniform distribution on $[0, 2\pi]$ to the uniform distribution on $[-1, 1]$, therefore the distribution of $\eta = \sin \xi$ is

$$p(\eta) = \begin{cases} \frac{1}{2} & \eta \in [-1, 1] \\ 0 & \text{else} \end{cases}$$